You Need a Map to Navigate a Teacher Education Programme: Analyzing the InformalMath Project through Conjecture Mapping

Serve una Mappa per Navigare un Programma di Formazione Insegnanti: L'analisi del Progetto InformalMath attraverso il Conjecture Mapping

Se Necesita un Mapa para Navegar un Programa De Formación de Profesores: Analizar el Proyecto InformalMath mediante el Conjecture Mapping

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Abstract. *This theoretical paper analyzes the InformalMath Project, emphasizing the implementation of Informal Mathematics Education (IME) within teacher education, particularly in the context of non-scientific museums. The project follows the approach of educational design research and makes use of the technique of conjecture mapping to structure the research process. By employing conjecture mapping, the study delineates the connection between teacher education programme characteristics for IME and the expected processes and outcomes. InformalMath stands as a pioneering endeavor in integrating IME principles into teacher education, within the realm of non-scientific museums. Through detailed analysis and the application of conjecture mapping, this paper lays foundational insights into the development of teacher education methodologies for IME.*

Keywords: informal mathematics education, teacher education, educational design research, conjecture mapping, non-scientific museums.

Sunto. *Questo articolo teorico analizza il progetto InformalMath, che lavora all'implementazione dell'Educazione Matematica Informale (EMI) nell'ambito della formazione degli insegnanti, in particolare nel contesto dei musei non scientifici. Il progetto segue l'approccio della educational design research e utilizza la tecnica del conjecture mapping per strutturare il processo di ricerca. Utilizzando il conjecture mapping, lo studio delinea la connessione tra le caratteristiche del programma di formazione degli insegnanti e i processi e i risultati attesi. InformalMath rappresenta un'impresa pionieristica nell'integrazione dei principi dell'EMI nella formazione degli insegnanti, nell'ambito di musei non scientifici. Attraverso un'analisi dettagliata e l'applicazione del conjecture mapping, il presente lavoro pone le basi per lo sviluppo* *di metodologie di formazione degli insegnanti nell'ambito dell'EMI.*

Parole chiave: educazione matematica informale, formazione insegnanti, educational design research, conjecture mapping, musei non scientifici.

Resumen. *Este artículo teórico analiza el proyecto InformalMath, haciendo hincapié en la implementación de la Educación Matemática Informal (EMI) dentro de la formación del profesorado, particularmente en el contexto de los museos no científicos. El proyecto sigue el enfoque de la educational design research y hace uso de la técnica del conjecture mapping para estructurar el proceso de investigación. Mediante el uso del conjecture mapping, el estudio delinea la conexión entre las características del programa de formación del profesorado y los procesos y resultados esperados. InformalMath es una iniciativa pionera en la integración de los principios de EMI en la formación del profesorado, en el ámbito de los museos no científicos. A través de un análisis detallado y de la aplicación del conjecture mapping, este artículo establece las bases para el desarrollo de metodologías de formación del profesorado para la EMI.*

Palabras clave: educación matemática informal, formación del profesorado, educational design research, conjecture mapping, museos no científicos.

1. Introduction

In this theoretical article, we introduce the InformalMath Project, giving a particular focus on the implementation of Informal Mathematics Education (IME) as described by Nemirovsky et al. (2017). IME refers to mathematics activities conducted outside the traditional school environment, yet within spaces intentionally designed to promote mathematical learning.

We characterize teacher education in IME in non-scientific museums as it unfolded during the project, and we show how its development laid the groundwork for the definition of the main characteristics of a teacher education programme promoting IME.

The project developed as an *educational design research* (McKenney & Reeves, 2019), a research approach that is characterized by the aim of solving problems in educational settings through educational interventions while also advancing theoretical understanding of the related phenomena.

In this case, on a theoretical level the research asks how teacher education can be worked on in the context of IME, from the point of view of intervention; on the other hand, it works on the design of professional teacher training from IME activities in non-scientific museums.

In order to clarify the connection between these two interrelated instances, we chose the approach of *conjecture mapping* which is "a systematic tool to visualize practice-based and theory-based goals (conjectures) and how these interrelate" (Deister et al., 2022, p. 2).

This paper aims to outline the project's research trajectory through conjecture mapping, clarifying the connections between the characteristics of the teacher education program for IME, the expectations for its implementation, and the anticipated outcomes. Given the significance of InformalMath as a pioneering endeavor, we anticipate further iterations of the programme: to facilitate these future iterations, a well-communicated theoretical foundation is required, which this paper seeks to establish through conjecture mapping.

Furthermore, in the broader context of IME educator training, it is essential to distinguish which aspects are specific to this programme and which are not. Here, too, conjecture mapping proves to be valuable. We aim to provide clarity on the programme contributions to the field of informal mathematics education, offering a roadmap for both theoretical exploration and practical application. This endeavor not only enhances our understanding of IME within the unique setting of non-scientific museums, but also contributes to the larger discourse on educators' preparation in this domain, helping to promote a vision of knowledge, and mathematics, as a cultural product.

2. Informal Mathematics Education

Over the last few decades, there has been growing interest in mathematics learning activities outside the traditional school setting. Various studies have reported how some students, despite a history of struggling with mathematics in the institutional context of school, have developed mathematical skills in work contexts. This phenomenon is exemplified in the studies by Nunes, Schliemann, and Carraher (Carraher et al., 1982; Nunes et al., 1993), which are part of the so-called *Street Mathematics* or *Everyday Mathematics* research strand. These studies highlight how young street vendors in Recife, Brazil, were able to easily solve arithmetic operations in a market context yet struggled with the same operations when presented in the written form typically used in schools. Similar findings have been reported in studies involving weavers in Mexico (Childs & Greenfiled, 1980) and tailors in Liberia (Lave, 1977). This line of research acknowledges a type of learning that occurs in an unintentional, contingent, and unplanned manner.

While this informs us that meaningful learning in mathematics takes place also in spaces different from formal classroom contexts, it also makes us reflect on the fact that it may be difficult to exploit the effectiveness of learning through everyday mathematics in formal contexts as well, precisely because of its characteristics of being unplanned, unintentional, and contingent. However, it is most likely that the integration of everyday mathematics approach in structured contexts can support formal learning, and perhaps even support the prevention of school failure, due to the inclusive characteristics it inherits from the contingent problem-solving approach. Calculating the price of a certain amount of goods or determining the most appropriate way to sew a dress, weave a carpet or a basket, and so on, can be engaging activities for people who are familiar with the context to which these activities belong and who are intrinsically motivated to solve a certain problem related to it. We refer to the learning that takes place in these contexts as *emergent learning* (Nemirovsky, 2018), where the term emergent has the meaning attributed to it in the theory of complex systems. In this field, it is used to denote certain behaviors of a system that, although well definable, are not easily predictable from the laws governing its components. Emergent learning – a learning "in which spontaneous memories, speculations, and projects of the participants may take center stage regardless of whether they accord with pre-conceived endpoints" (Nemirovsky, p. 403) – always occurs, unlike the learning for a certain purpose defined a priori, which can therefore fail or succeed. Nevertheless, emergent learning can occur according to trajectories that are difficult to predict a priori.

One hallmark of emergent learning pedagogies is their capacity to prioritize participants' spontaneous memories, thoughts, and projects, even when these do not align with preset objectives. Due to their distinctive openness to unexpected directions, absence of predetermined and measurable outcomes, and largely optional participation, these pedagogies face challenges in formal educational settings. However, it is precisely these attributes that render them particularly valuable in the context of "informal mathematics education" (p. 418).

In a chapter of the 'Futuristic Issues' section in the *Compendium for Research in Mathematics Education*, Nemirovsky, Kelton, and Civil characterize IME as occurring in environments that, unlike street or everyday Mathematics, are "intentionally designed to support mathematics learning, whether because they are structured through programs with regular schedules and assigned educators or because they host technologies, tools, or exhibits designed to engage the user with mathematics" (Nemirovsky et al., 2017, p. 970). Furthermore, IME is distinguished from traditional classroom mathematics for three primary reasons:

- (IME-1). The learners' free choice: "for the most part, learners volunteer to participate in them or are relatively free to pursue their own interests once they are in the environment";
- (IME-2). The fluidity of the boundaries between disciplines: "activities may drift from mathematics to art, literature, science, games, technology, and so forth";
- (IME-3). The absence of traditional forms of academic assessment: "Informal mathematics education needs to be documented for the purposes of professional development and collective exchange, but learners are not individually graded with scores" (Nemirovsky et al., 2017, p. 970).

Although the characterization of IME is drawn by difference from street mathematics and classroom mathematics, the goal of IME is not to resolve the disparities between in-school and out-of-school mathematics. Instead, it seeks to cultivate social environments where the mathematical engagement is not rigidly defined by established curricula or textbooks. This approach allows for a more open and creative space, where participants are free to draw from their memories, innovate, make connections, or express their emotions, building a space where emergent learning not only can occur, but is recognized and valued.

In the next section, we provide an example of an IME activity within the InformalMath project (for other examples see Nemirovsky et al., 2017; Kelton & Nemirovsky, 2023).

3. Informal Mathematics Education Workshops in Non-Scientific Museums

Building upon the research foundations of IME and the intricacies of learning outside classrooms, the project InformalMath focuses on applying IME within the distinctive setting of non-scientific museums, where the term 'non-scientific museums' refers to museums beyond those dedicated to science and technology, where the relevance to mathematics may not be as immediately apparent. As we have seen, this specific context allows for a detailed exploration of how IME principles can be integrated into the educational initiatives of cultural institutions, influencing learning experiences and pedagogical methods.

The concept of tapping into the potential of non-scientific museums arose from the participation of the first author in the Next-Land project (www.next-level.it/progetti/next-land-2/). He contributed alongside a team of researchers to the design of IME activities,[1] like the one presented in this section, in four art and history museums located in Turin, Italy (Casi et al., 2022). These activities were aimed at sixth- and seventh-grade students, and were attended by more than 300 students, accompanied by their teachers.

As we have emphasized, Informal Mathematics Education Workshops (IMEW) are crafted in alignment with IME principles, but they are uniquely structured through a robust laboratory character and a particular emphasis on the role of artifacts in both teaching and learning mathematics. The mathematics laboratory is a didactical methodology extensively explored within the Italian research domain of mathematics education (see, for example, Arzarello & Robutti, 2008), that has been described by Anichini et al. (2004) as encompassing not just a physical space, but also a dynamic learning environment where students are proactive, formulate and test hypotheses, design and conduct experiments, engage in discussion to justify their choices, learn to gather data, negotiate and construct meaning, and steer the development of personal and collective knowledge towards both temporary conclusions and new inquiries (p. 49, translated by the authors). These widely studied topics lend themselves to be exploited both in the context of IME and in the context of the

[1] The designers team was composed by Raffaele Casi, Valentina Leo, and Chiara Pizzarelli, under the scientific direction provided by Cristina Sabena.

classroom. Their value in the classroom is well known, but we cannot overlook how the attitude of discovery and the formulation and justification of hypotheses typical of the laboratory leave room for unexpected learning trajectories, fertile ground for the emergent learning valorized in IME, and for a fluidity of boundaries between disciplines.

To further illustrate the concept of IME, let us examine a specific example drawn from the work of the first author (Casi et al, 2022). We present an IMEW (Casi, 2023) designed for the National Museum of the Italian Risorgimento, situated in the historic Palazzo Carignano in Turin, Italy. This is a historical museum that traces the stages of the political unification of the Italian state, starting from the end of the 18th century, up to the birth of the Italian Republic in 1946. The IMEW under scrutiny, titled "Freedom Shall Decrypt" ("Libertà Va' Decrittando", in Italian), draws inspiration from the museum collection. It particularly revolves around the "Cavour's cipher", a significant artifact used for classified communications between King Vittorio Emanuele II di Savoia and his Prime Minister, Camillo Benso Conte di Cavour. This cipher acts as a portal to the fascinating world of cryptography, offering participants a blend of guided tour and workshop. Participants delve into the art of decryption, employing various methods and artifacts to decipher messages, thereby unlocking the narrative of the museum. Techniques include the use of substitution ciphers, such as the "Carbonaro code", Caesar's cipher, and Leon Battista Alberti's disk, alongside steganography and transposition methods like the Spartan scytale. Through decrypting each code, participants reconstruct the critical moments that led to the creation of the Italian state, merging mathematical exploration with historical discovery.

The example of the "Freedom Shall Decrypt" IMEW at the National Museum of the Italian Risorgimento aligns with the criteria for IME as outlined by Nemirovsky et al. (2017). This IMEW is in fact intentionally designed to support mathematical learning within a museum environment, leveraging artifacts and historical narratives to engage participants with mathematics. It incorporates structured activities facilitated by the exhibits and the artifacts of the museum, specifically tailored to explore mathematical concepts through the lens of history, providing a comprehensive and engaging experience that transcends the traditional classroom environment.

During the implementation of the IMEW with the students and teachers involved in the Next-Land project we could observe a general appreciation for the quality of the activities, the richness of the insights provided and the involvement that occurred. Nevertheless, we could not avoid highlighting a potential pitfall: if IME activities, offered as school field trips, are not effectively integrated into the classroom curriculum by teachers, they risk becoming a 'firework' for the students—a momentary delight that swiftly dissipates, leaving only a faint memory and perhaps a tinge of nostalgia.

Driven by the desire to circumvent the transient firework effect and recognizing that a meaningful connection with everyday classroom experiences can enhance the significance of students' experiences during IMEWs, in the autumn of 2021, the first author and Cristina Sabena had the idea of working on teacher education to promote learning that was both related to the design of IMEWs,[2] and to the professional development of teachers in their everyday activities (Casi & Sabena, 2022). This decision stems from the understanding that laboratory-based teaching and the use of artifacts are known to be effective for learning mathematics within school contexts. Furthermore, we assume, following the result related to both IME and street mathematics (Nemirovsky et al., 2017; Nunes et al., 1993), that integrating IMEW into structured contexts can bolster formal learning and potentially help prevent school failure, thanks to its inclusive nature and emphasis on contingent problem-solving. As we have highlighted in this section, professional development on IMEWs aims to promote active, laboratory-based teaching with a focus on artifacts, while also fostering emergent learning in mathematics.

We can now move on to introduce the aim of this paper and the suestions in the following section.

4. Research Questions and Aim

Given the innovative nature of IME activities, especially in the Italian context, there was a lack of a substantial research foundation to underpin the teacher education programme (refer to Carotenuto et al., 2020 for an example). Moreover, Nemirovsky and colleagues (2017) emphasized the importance of investigating training of IME educators: "To the extent that informal education practices differ qualitatively from formal education ones, it is clear that the education of informal mathematics educators needs to follow approaches different from prevalent ones in mathematics teacher education" (Nemirovsky et al., 2017, p. 977).

For this reason, InformalMath is a dual-purpose project: for what concerns teacher education, it is a comprehensive two-year program designed for primary and middle school educators. Its goal is to introduce these teachers to IME and to actively involve them in this area by designing IMEWs in non-scientific museums. From a research standpoint, the project seeks to establish foundational guidelines for creating IME teacher education courses addressing the issue highlighted by Nemirovsky and colleagues (2017).

The second purpose can be further split into two major objectives. Firstly, it aims to lay the foundation for understanding how to implement IME with teachers within non-scientific museums. Secondly, it seeks to develop and

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[2] InformalMath was conceptualised, designed and realised by Raffaele Casi and Cristina Sabena, as part of Raffaele Casi's PhD project, under the supervision of Prof. Cristina Sabena. See the website of the project at www.informalmath.unito.it

define the educational methodologies for IME educators. This contributes to the broader research discussion on "vibrant and socially significant IME", marking relevant progress in the field.

In this paper, we focus specifically on the second objective: the definition of educational methodologies for IME educators, and in particular on the definition and foundation of expected processes and outcomes in relation to the characteristics of the teacher education programme InformalMath. As a matter of fact, our research questions (RQs) are the following:

RQ1. How can the expected processes and outcomes of InformalMath be outlined in relation to the development of the teacher education programme?

and, consequently,

RQ2. What foundational elements are crucial for the development of IME teacher education methodologies in the context of non-scientific museums with respect to the expected outcomes?

To help the reader become acquainted with the complexity of the programme and the innovative nature of the topics it covers, and to ensure a comprehensive understanding of its application within the research context, we briefly present and discuss the InformalMath teacher education programme in the next section.

5. InformalMath Teacher Education Programme

The InformalMath programme commenced in December 2021 and was concluded in May 2024. In the programme, the participation to and the development of new IMEWs was used both as a tool and a goal for teacher education, consistently with the first purpose of the project. As a tool, the process of designing new workshops with the teachers provided them with an opportunity to delve into the fundamental concepts of IME. As a goal, their design efforts equipped the museums—and thereby the students in the community—with fresh IMEWs to explore.

The programme was organized into three distinct phases: Phase 1 – Discovery, Phase 2 – Guided Co-Design, and Phase 3 – Scaffolded Co-Design. The programme was designed to progressively guide participating teachers who voluntarily joined the project towards co-design of IMEWs. This approach was inspired by the cognitive apprenticeship model (Collins et al., 1989), which begins with teacher educators playing a significant role that gradually diminishes as teachers increasingly assume more responsibility and autonomy as the programme advances.

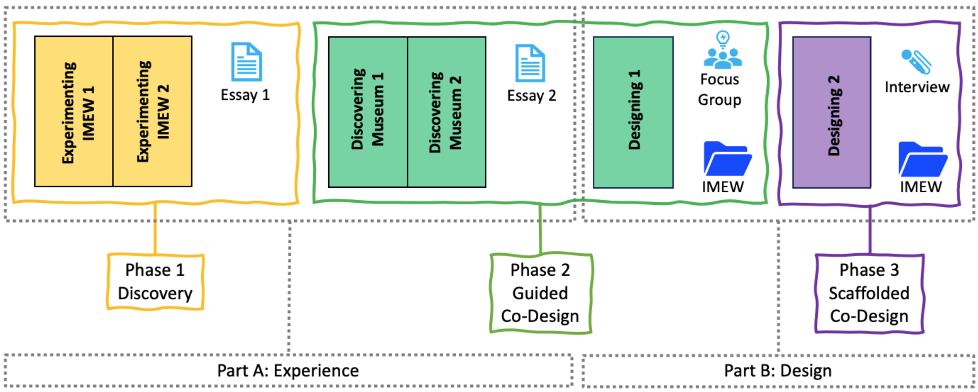
The choice to involve teachers on a voluntary basis was made in accordance with principle IME-1 reported in Section 2, concerning the freedom of choice of participants in informal mathematics education activities. At the same time, principles IME-2, concerning the fluidity of boundaries between disciplines, and IME-3, concerning the absence of traditional forms of evaluation for participating teachers, were also followed. The set of choices thus made InformalMath an informal mathematics education programme for teachers.

Twenty-seven teachers joined the project in Phase 1, of whom 22 continued in Phase 2, and 7 concluded with Phase 3. Such an evident decrease in the number of participants, especially between Phase 2 and Phase 3, can be explained by the long duration – over two and a half years – of the teacher education programme. Indeed, many of the teachers who chose to leave the project reported that they made this choice as a result of personal life changes, such as pregnancies or relocations, or professional changes, such as school or career changes, or because of the demand to take a more central role in school administration, thus having to give up committing considerable time to the design of the IMEWs.

Figure 1 presents a detailed diagram that outlines the organizational structure and main components of the programme.

Figure 1

The Structure of the InformalMath Teacher Education Programme [own figure]



The programme is ideally divided into two parts: part A and part B. Part A is characterized by teachers making experience of IMEWs, of museums, and deepening of mathematical and pedagogical themes. Part B is characterized by the teacher's design of new IMEWs. More specifically, during phase 1 (December 2021 – March 2022), the teachers experienced two of the previously planned IMEWs: 'Freedom Shall Decrypt' at Museo Nazionale del Risorgimento and 'Swirls of Ideas' (Vortici di idee, in Italian), an IMEW on the topic of spirals at the City Museum of Ancient Art, hosted in Palazzo Madama, Turin (for a detailed description of this IMEW see Casi et al., 2022). For two days, two similar activities were proposed, with this structure: experience of the IMEW, personal reflection, discussion, and in-depth study in mathematics and mathematics education. At the end of phase 1, teachers were asked to produce a first essay reviewing the experience: the choice to collect teachers' voices, which characterizes the entire programme, is consistent with emerging learning pedagogies that, as highlighted in Section 2, prioritize participants' voices, even when these do not align with preset objectives.

Phase 2 took place from April 2022 to January 2023 and marks the transition from part A of the course to part B: in the following we describe its sub-phases as phase 2A and phase 2B. Phase 2A aimed at providing teachers with additional tools for the design of new IMEWs. It followed phase 1, with some differences: in two days the teachers first visited two museums (PAV – Parc of Living Art in Turin and Castello di Rivoli Museum of Contemporary Art in Rivoli), then reflected and discussed about the museum's collections and their use for IME and participated in in-depth discussions in mathematics education and pedagogy. At the end of Phase 2A the teachers produced a second essay reviewing the experience. Phase 2B launched the planning in groups, organized along the pattern of successive refinements following the feedback each group received from the teacher educators, other participants, and experienced museum staff. At the end of phase 2, focus groups were organized to review the experience, and the groups delivered the first set of designs.

Phase 3 (February 2023 – May 2024) had a structure similar to the one of phase 2B, with the fundamental difference of a preliminary step in which the teachers were asked to select the museums for which to design IMEWs, giving more agency in the designing process to the teachers. The choice was made for the 'Soundscape Museum' (Museo del Paesaggio Sonoro) in Riva Presso Chieri and the Castle of Moncalieri – Sabaudian residence. Also, at the end of phase 3, the teachers were given the opportunity to review their experience through an individual interview, and the groups delivered the second set of designs.

Now that we have introduced the main elements of the teacher education programme InformalMath, and the objectives behind it, we can describe the research approach behind the InformalMath programme, that of Educational Design Research (EDR): this will help to understand how the research process developed to define the founding characteristics of the teacher education programme and the expectations with respect to these characteristics.

6. InformalMath as an Educational Design Research

The approach of EDR encompasses a broad spectrum of investigations aimed at addressing critical issues within actual educational contexts, whilst simultaneously contributing to theoretical understanding regarding significant phenomena (McKenney & Reeves, 2019; Bakker, 2018). In this case, the addressed issue is the lack of professional development in relation to IME. This need can be found in existing literature on IME (Nemirovsky et al.,2017), but it arose also in the context of the Next-Land project mentioned in Section 3, that promoted the participation of numerous students and teachers in IMEWs, who subsequently showed an interest in learning more about IME.

EDR is characterized by its methodological flexibility: quantitative, qualitative, and mixed-method approaches can be used during the research process (McKenney & Reeves, 2019). Due to a lack of research on the topic of professional development on IME, the present work has an explorative and fundative aim: for this reason it mainly adopts a qualitative approach. A defining feature of EDR is the active involvement of educational practitioners not merely as recipients of research outcomes, but as co-contributors to the cyclical process of design, implementation, and assessment. For what concern the structuring of InformalMath programme the educational practitioners involved were the researchers, in the role of teacher educators, the museum experts, and the participating teachers. The characteristics of the programme have been defined through a work of reflection and analysis that involved all these voices along the way, collected through essays, interviews, or focus groups, as described in Section 5.

We delineate the core attributes of EDR as identified by McKenney & Reeves (2019), and we establish their relevance to the InformalMath project. EDR is distinguished by its:

- Theoretical Orientation: EDR is initiated within a theoretical framework, from which the educational intervention or design is structured. This design is developed and modified coherently, enriching or expanding the initial theory, or enhancing the theoretical understanding of the investigated phenomenon through the reevaluation of various theoretical constructs. Theoretical understanding is augmented not only by empirical results but also by how these results influence the proposed design. From this point of view, we have seen how the InformalMath project attempts to establish guidelines for the creation of IME teacher education courses in non-scientific museums.
- Interventionist Nature: The design proposed in an EDR aims to impact the resolution of problems arising in educational contexts, whether formal or informal. EDR seeks solutions to specific problems, presenting these solutions in the form of design principles related to educational materials, instructional models, professional development pathways, etc. From this perspective, InformalMath is a programme designed for school educators. Its aim is to introduce these teachers to IME and to actively involve them in this area.
- Collaborative Approach: Addressing problems within the complex systems of educational environments necessitates the coordination of various disciplines and professional roles, as well as integrating different perspectives. This ensures that the research process engages multiple participants over time, starting with all those for whom the addressed problem is relevant. To make an example we can refer to the work of Casi and Sabena (2023), where it is shown that the analysis of teachers' reflection about one of the phases of the InformalMath programme significantly

shaped the subsequent phases and the professional development.

- Responsively Grounded: Throughout the research process, advancements may occur on both practical and theoretical levels, necessitating a transformation in the adopted perspective. New information from literature, field interventions, or other sources is processed and utilized to inform subsequent decisions. In the presented project the different actors involved in the research indicated, in successive stages, what the founding elements of the pathway could be: as will be seen below, for example, the structured exchange of feedback between teachers, museum experts, and didactic experts enabled the design of IMEW that were the result of a collective contribution.
- Iterative Process: Due to its responsively grounded nature, EDR is structured around cycles of design, implementation, and evaluation that increasingly involve more participants and greater detail. Each EDR is divided into cycles and sub-cycles, each with distinct objectives and characteristics. We can recognize in the InformalMath programme the first macro-cycle of research, where design principles of the programme are defined and analyzed in a first endeavor of formalizing the main characteristics of a professional development on IME (Figure 2). We describe below the interplay of micro-cycle and meso-cycle in the project.

Figure 2
The Structure of InformalMath as a Research Project [own figure]

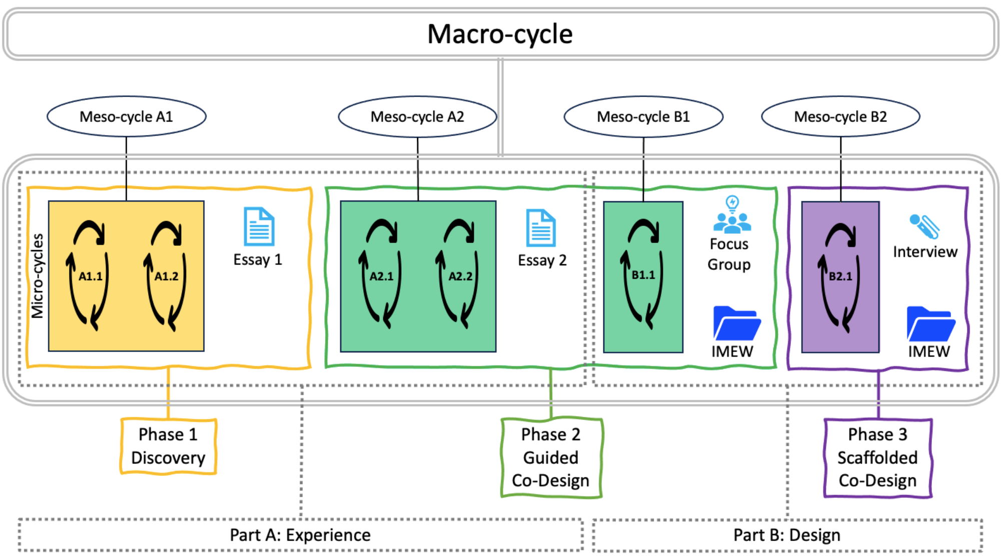


Figure 2 supplements Figure 1 by offering an insight into the research process of the InformalMath project. It illustrates the experiences of teachers in Part A as consisting of four micro-cycles. Each cycle shares a similar format, as detailed previously: a museum visit and immediate impression gathering in the morning collected through an online form, followed by a discussion of these impressions and two in-depth discussions in the afternoon. In micro-cycles A1.1 and A1.2, the museum visits were conducted through IMEW, and the in-depth discussion focused on specific mathematical themes (cryptography and cryptanalysis for A1.1, and spirals along with geometric transformations for A1.2) and mathematics education topics (problem-solving and posing for A1.1, and semiotic mediation of artifacts for A1.2). For A2.1 and A2.2, traditional guided museum tours were conducted, and the in-depth discussion gave insights into mathematics education (orientation and spatial representation for A2.1, and didactic transposition for A2.2) and pedagogical approaches (educational activity planning for A2.1, and multidisciplinary approaches for A2.2).

Part B features two micro-cycles, B1.1 and B2.1, focused on the design of IMEWs, characterized by a consistent structure of small-group design, feedback collection, redesign, presentation to museum experts and other participants for additional feedback, and final design. A distinctive step in micro-cycle B2.1 was each design group selecting a local museum for their IMEW design.

At a meso level, four meso-cycles are identified: A1 and A2 in Part A, and B1 and B2 in Part B, each incorporating the described micro-cycles. A key aspect of each meso-cycle was the collection of participants' voices, where teachers were encouraged to reflect on their experiences, discussing challenges, opportunities, and learnings. Teachers' voice was collected in written form at the end of A1 and A2 through personal essays, and verbally at the end of B1 and B2, using focus groups and individual interviews, respectively. This collection served dual purposes: it informed theoretical understanding by expanding knowledge on teacher education processes and enhanced practical applications by offering insights for designing future meso-cycles.

As discussed before, the entire InformalMath programme is the macro-cycle, which has not been repeated so far, and which constitutes the research subject discussed in this paper.

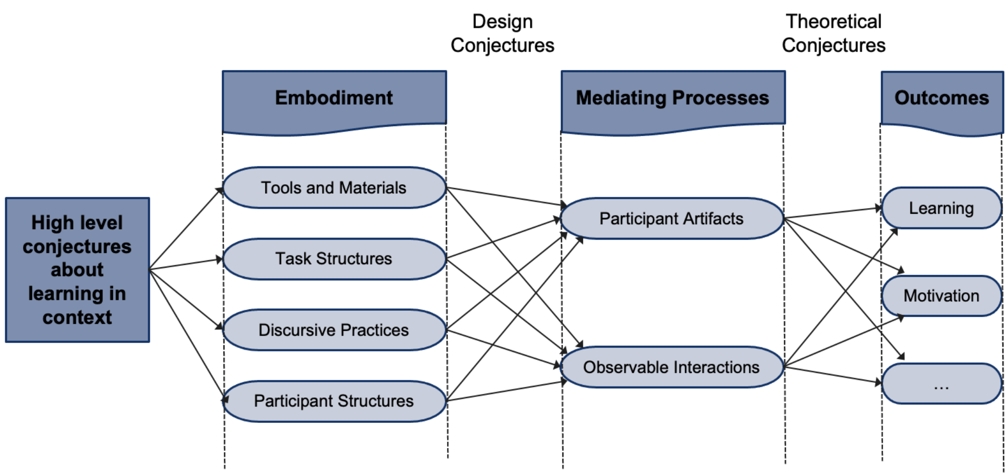
In order to trace the main elements of the macro-cycle of InformalMath research, it was decided to use a technique known in EDR, that of conjecture mapping. The final revision of the programme was conducted using the conjecture map, prompted by the general need to systematize the approach and ensure its reproducibility in different contexts from that of the first author, and without their supervision. This technique is presented in detail in the following section.

7. Conjecture Mapping in Educational Design Research

Theoretical development and the evolution of interventions in EDR are achieved through iterative cycles of analysis, design, implementation, and evaluation. EDR is thus not merely a methodology, but a genre of research that connects design with associated learning hypotheses, which are explored and validated through the implementation of the design. In other words, in an EDR it is imperative to clarify the relationships between the design and the expected processes, as well as between these processes and the expected outcomes. Sandoval (2014) defines these two kinds of relationships as *design conjectures* and *theoretical conjectures*, respectively. To facilitate the organization of research around these conjectures, Sandoval advocates for the use of the conjecture map, highlighting its role in delineating the foundational structure of the project. Conjecture maps are crafted to delineate hypotheses connecting design features (*embodiment*), *mediating processes*, and the resultant *outcomes* within educational contexts. The process of defining a conjecture map renders the theorization about the interconnections among these elements more transparent. For this reason, it serves not merely as a means of representation, but also as instruments for investigation and reflection. It compels researchers to specify not only the objectives of their design endeavors but also to articulate the anticipated functions of specific design features, how these features are expected to interact, and the outcomes they are intended to generate (ibid.). Conjecture mapping technique has been widely used in educational domains, as well as in mathematics education (e.g. Boalens et al. 2020; Deister et al., 2022; Choppin et al., 2018).

In Figure 3 we present the general structure of a conjecture map as introduced by Sandoval (2014).

Figure 3
General Structure of a Conjecture Map for Educational Design Research [adapted from Sandoval (2014, p. 1), own figure]



The framework of a conjecture map is structured in four columns. The first one is that of *high-level conjectures*. These conjectures are grounded in theoretical understanding of how learning can be supported within a specific context. In this case we will use these conjectures to frame the InformalMath programme. The next column, that of *embodiment*, involves the materialization of the high-level conjectures into tangible design elements. Sandoval proposes four primary categories for *embodiment* within learning environments: tools and materials (such as instruments and resources), task structures (including goals and criteria for tasks), participant structures (defining various roles and responsibilities), and discursive practices related to communication during the intervention. Sandoval himself highlights how these categories are not necessarily present in all cases. The implementation of these design features triggers *mediating processes* (third column), which the author suggests examining through two lenses: as observable interactions within the designed environment and as artifacts created by participants during the intervention. Mediating processes are a key element in the definition of the relation between embodiment and *outcomes* (last column). Sandoval proposes some examples of common outcome categories referring to learning, motivation, etc. *Theoretical conjectures* are the links between mediating processes and expected outcomes, while *design conjectures* structure the relation between embodiment and mediating processes. Sandoval further discusses the challenges in directly correlating mediating processes with learning outcomes, due to the intricate nature of educational practices. Echoing Salomon (1996), he advises against viewing conjecture maps as linear causation models, but rather as frameworks outlining process relationships and patterns of change.

The second author, in one of her previous works (Deister et al., 2022) highlighted how conjecture mapping is not a research method, but a technique for structuring work. The work of Deister and colleagues lists several uses of this technique in EDR, which can be found in the literature: here we refer to two of them. First, the present work wishes to introduce conjecture map to make explicit what processes are expected during the implementation of an educational intervention, highlighting the link with the intervention itself. Secondly, the conjecture map of the project aims to support clarity in the communication of the main assumptions behind the research, supporting the structuring of successive iterations of the intervention, even in contexts different from the initial one.

Furthermore, the map in InformalMath project is intended to link different findings belonging to different phases of the training initiative. The purpose is to maintain coherence in the interpretation and analysis of the data according to the objectives of the educational intervention, that is built over a long period and several phases. Since the conjecture map supports the structuring of the link between course characteristics, processes and expected outcomes, it will allow us to answer the research question RQ1.

In the following section we define the map of the project. We initially focus on the High-Level Conjectures, before moving on to discuss the column related to embodiment, which outlines the main features of the intervention. Subsequently, we describe the remaining columns. Finally, we discuss the design and theoretical conjectures upon which the project is built. This discussion aims to define the foundational elements of IME teacher education methodologies within the context of non-scientific museums, addressing research question RQ2.

8. InformalMath Conjecture Map

We start building the conjecture map related to the InformalMath programme defining the high-level conjecture and showing how this conjecture shapes the embodiment of the intervention. At the end of Section 3 we highlighted how the starting assumption, that we can now call high level conjecture, of the project is that experimenting with and designing IMEWs enables teachers to develop professionalizing skills that can be used in everyday teaching practice: on the one hand because it allows the encounter with educational methodologies considered significant for the teaching of mathematics in the classroom, such as the laboratory (see Section 3), and on the other hand because IME perspective allows to cultivate social environments where the mathematical engagement happens in an open and creative space. We described the main phases of the professional development designed in Section 5, and in Section 6 we highlighted how they relate to different research cycles.

Here we come back again to the structure of the InformalMath programme in order to highlight the characteristics of each phase and structure the embodiment column of the conjecture map. To do so, we summarize the main aspects of the programme and categorize them according to the elements proposed by Sandoval in his original work (2014):

- Task structure:
  - Phase 1: Teachers experience IMEW as students, then discuss the experience and participate in two in-depth discussions, one with a focus on the mathematics and one with a focus on mathematics education;
  - Phase 2: Teachers visit the museum with a focus on planning, then discuss the experience and participate in two in-depth discussions, one on mathematics education and one on pedagogy; later engage in the design work, with exchange of feedback between colleagues, with MathEducation experts and with museum experts;
  - Phase 3: Phase characterized by design work, with exchange of feedback between colleagues, with MathEducation experts and with museum experts.
- Participant structure:
  - Phase 1: Teachers have the role of students: they are active, explore, use the available artifacts working in groups;

  - Phase 2A: Teachers no longer play the role of the students, but explore the museum space with a view to future planning;
  - Phase 2B and 3: Teachers work on the design in small groups, there is a structured feedback exchange between teachers, museum experts, and mathematics education experts.
- Discursive practices: the key feature with regard to this aspect is the choice to give voice to teachers and their perspective at each stage, hence alternating moments of individual and collective reflection, using the following techniques:
  - a short paragraph filled with heartfelt impressions from the experiences;
  - reflective essays composed with objective, analytical insights;
  - focus groups and interviews centered on reflective discussions about design actions.

8.1. Definition of Mediating Processes and Design Conjectures

We can now link each element of the column of the embodiment, described in the previous section, to specific mediating processes. Each association between these two elements, thus between the two columns of embodiment and mediating processes, represents a *design conjecture*. Figure 4 summarizes the elements of the conjecture map described so far: each arrow represents the construction of a design conjecture.

Mediating processes are associated with the active participation of the teacher in interactions with other teachers or artifacts during the workshop or in other phases of the programme: as indicated in Figure 4, this type of activation is expected when the teacher is put in a position to carry out activities with others or with specific artifacts, whether related to participation in the IMEW as a student or to group design with other teachers.

The written reflections, that are part of the embodiment, allow us to observe the shift towards reference to ongoing mathematical processes rather than to objects or topics. Reference to the role of artifacts in the activity or design is also observed in the texts produced. The arrows in Figure 4 show that references to mathematical processes rather than objects are expected mainly in relation to the teachers' experience of IMEW as learners, i.e. when they put themselves in a position to learn as students. The reference to the role of artifacts is instead expected when teachers have to design IMEWs in museums, reflecting precisely on the relationship of artifacts between museum and mathematical knowledge.

Two other elements sought in individual reflections are the reference to the students and their role in the IMEW, which can be associated with learning possibilities, affective factors, involvement and motivation, etc., and the reference to interactions with the museum as a space that educates. The latter aspect is also sought in the designs produced during the course.

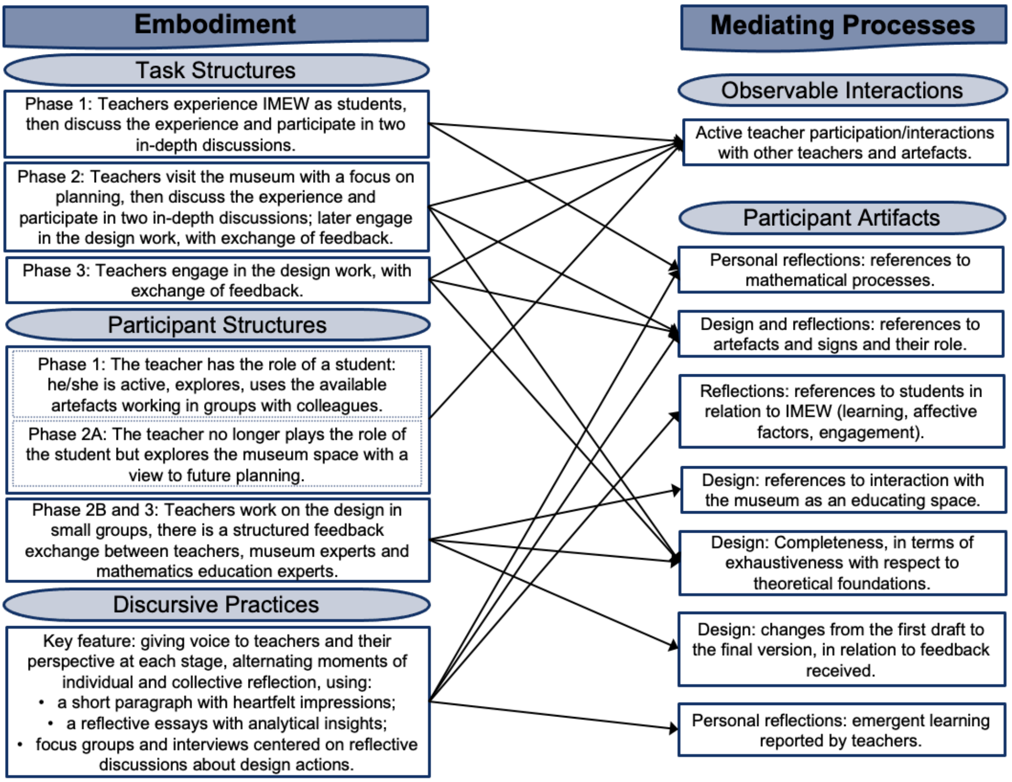
With regard to the IMEW designs produced, completeness is noted with respect to the interaction with the museum, and with the artifacts in play, the activation of mathematical processes, and the description of the phases and working methods considered.

Another relevant aspect in relation to the exchange of feedback are the changes implemented from the first draft to the final version of the IMEW designs, in relation to the feedback received both from colleagues, mathematics education experts, and museum experts.

A transversal aspect, fundamental when working in an IME perspective, are the emergent and unforeseen learnings resulting from the activity performed: for this aspect, reference is made to those learnings explicitly indicated by teachers in personal reflections.

Figure 4

Embodiment and Mediating Processes of InformalMath [own figure]



8.2. Expected Outcomes and Theoretical Conjectures

In this section we present the expected outcomes of the InformalMath programme and we link them to the embodiment through the mediating processes: Figure 5 shows the result of this work.

One of the main general objectives of the programme is to ensure that participants see knowledge, and thus also mathematics, as a cultural product.

Moreover, in the course we work in the direction of promoting the teachers' use of mathematical education experiences as an opportunity for disorientation, with a focus on activating an exploratory approach in students. This is not only in the context of the IMEW, but in general in the classroom context.

If we look at the specific objectives, autonomy in the planning of IMEW is one of the main ones, together with the ability to collaborate with different professional figures with a focus on planning educational activities. As stated in the previous section, working from an IME perspective, it is important to give space to all emergent learning that can appear during the process and that could have not been expected.

Let us make explicit the link between the embodiment, the mediating process, and the first of the outcome addressed: It is the reference to mathematical processes in personal reflections, but also to artifacts and signs and their role in design that is, in the project, associated with a vision of mathematics as a cultural product (see Figure 5). In fact, we move from considering mathematics as a set of static elements to considering it as a set of behaviors, processes, and actions that have crystallized over time, and which can change according to the historical and cultural context of reference.

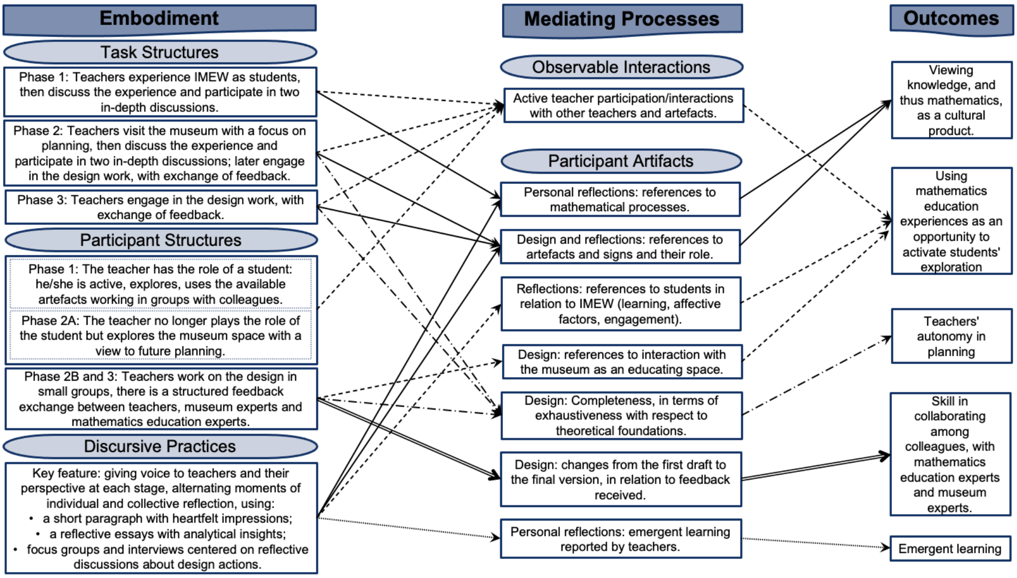
Conversely, considering the role of mathematics education in fostering an exploratory mindset among students, the design phase (described in the embodiment column in Figure 5) emerges as a critical juncture. In this phase, the museum is transformed into a learning environment where there is a focus on creating contextualized activities, and references to these aspects can be found in the teachers' reflections. These activities are structured like workshops and are defined by the deliberate choice of artifacts that enhance the intended mathematical experience. This phase is crucial for establishing a connection between regular school experiences and IMEWs: teachers referring to students during reflective phases, as stated in the mediating process column in Figure 5, highlight the formation of this type of connection, not just in relation to the teacher's personal experience. Another important aspect in supporting students' engagement is the activation experience of the participating teacher. In this context, interactions with others and with artifacts become an indicator, or observable process, of this outcome.

Teachers' autonomy in designing and their ability to collaborate can be identified in the finished products, hence in the designs that the teacher achieves at the end of the course.

It is now possible to complete the conjecture map of the project in Figure 5, and to highlight the trajectories that characterize the research pathway altogether. In order to improve the visibility of the map, we have chosen not to include the starting high-level conjecture, which is described at the beginning of this section.

Figure 5

Conjecture Map of InformalMath [own figure]



9. Discussion

An overview of the map shows how each characteristic of the teacher education programme for IME displayed in the embodiment column is linked to certain (mediating) processes in the programme implementation, which in turn are associated with the expected outcomes. These relationships derive from the theoretical research work presented in this paper, from the programme implementation, and from the analysis of the materials produced, such as, written essays, interviews, and focus groups (Casi & Sabena, 2023; Casi & Sabena, 2024).

As a result, it is possible by navigating the map in Figure 5 to answer research question RQ1 and to lay the foundation for subsequent programme implementations.

Upon closer examination of the map, and adopting a procedure similar to that of Boelens and colleagues (2020), it becomes evident that, starting from the outcomes and tracing back towards the left side of the image, one can identify five distinct trajectories, the tracking of which is facilitated by the different arrow patterns. These trajectories enable us to map the connections between the outcomes and the characteristics of the programme, through specific observables – the mediating processes. These trajectories lay the groundwork for the design and execution of future experimental iterations of the InformalMath macro-cycle. Moreover, they establish, for each phase, the core elements of IME teacher education methodologies in the context of non-scientific museums, thereby addressing RQ2. Now, let’s briefly revisit these five trajectories, resumed in Table 1.

Table 1

Trajectories Highlighted in InformalMath by Conjecture Mapping

Outcomes	**Mediating processes**	**Embodiment**
α. Viewing knowledge, and thus mathematics, as a cultural product.	1. References to mathematical processes in teachers' personal reflections. 2. References to artifacts and signs and their role in designs and in personal reflections.	a. (Phases 1 and 2): Visiting experiences in the museums, engaging in in-depth discussions with experts. b. (Phases 2 and 3): Designing IMEWs collaboratively.
β. Using mathematics education experiences as an opportunity to activate students' exploration.	3. Active teacher participation / interactions with other teachers and artifacts. 4. References to students in relation to IMEW in terms of learning, affective factors, and engagement in personal and collective reflections. 5. References to interaction with the museum as an educating space.	c. (Phases 1 and 2): Visiting experiences in the museums, first as students then with a designer's eye, engaging in in-depth discussions with experts. d. (Phases 2 and 3): Designing IMEWs collaboratively, through exchanging feedback and redesigning. e. Reflecting, sharing and discussing their reflections by using different techniques.
γ. Teachers' autonomy in planning.	6. Completeness of the produced designs, in terms of exhaustiveness with respect to theoretical foundations.	f. Visiting museums as IMEWs designers and engaging in in-depth discussions with experts. g. Designing collaboratively IMEWs with exchange of feedback.
δ. Skill in collaborating among colleagues and with experts.	7. Changes from the first draft of the design to the final version, in relation to feedback received.	h. Designing collaboratively IMEWs with exchange of feedback.
ε. Emergent learning.	8. Emergent learning reported by teachers in personal reflections.	i. Reflecting, sharing and discussing their reflections by using different techniques.

We highlight once more the significant role played by the voices of various participants throughout the research process: the mediating processes are defined including insights from museum experts and mathematics education experts, but most importantly, the perspectives of teachers at different stages.

This aspect is relevant in the context of IME: in a context of voluntary participation (IME-1), in which there are no forms of traditional academic assessment (IME-3) and in which emergent learning is to be valued along with expected learning, the participants' voices become a fundamental element for the course designer and for the understanding of its functioning.

Each row in Table 1 shows the identified trajectories that we can read as follow, laying the foundations for the design principles of InformalMath in the form of heuristic statements as proposed by Van den Akker (1999):

Within the context of IME teacher education programme in non-scientific museums, in order to achieve the outcome(s) {α, ..., ε}, observed through the mediating process(es) {1, ..., 8} [and {1, ..., 8}, and {1, ..., 8}], the programme can be structured by {a, ..., i} [and {a, ..., i}, and {a, ..., i}].

As in Nemirovsky's work (2018), we are far from trying to demonstrate "best practices" in IME educators training. We have shared, in an organized form through conjecture map, some experiences within the InformalMath programme, reading them in the light of the considerations on pedagogies of emergent learning.

> For this kind of pedagogy there are no best practices because no concrete attempt can be isolated from the circumstances of its development, the contingencies pervading its daily events, and the life history of the participant individuals and institutions. At most, given historic and contextual aspects, one can discriminate promising or rather-to-be-avoided ways of doing things. (Nemirovsky, 2018, p. 418).

If, as we have already mentioned, the map and the resulting Table 1, are two tools that on the one hand help us to identify what practices might characterize the specific InformalMath teacher education programme, on the other hand they answer question RQ2.

These tools spotlight "promising experiences" or "promising ways" which can be considered in designing teacher education initiatives within non-scientific museum contexts. Indeed, with the aim of achieving identical outcomes {α, …, ε}, observed through the same mediating processes {1, …, 8}, a feasible design strategy would involve adopting the characteristics defined for the InformalMath framework {a, …, i} and adapting them in a manner consistent with the mediating processes {1, …, 8}.

For example, if we consider outcome α, linked to the view of mathematics as a cultural product, it becomes characteristic for an IME approach in teacher education. We can work on the identification of such outcome from the analysis of specific elements in the teachers' written productions (mediating processes 1 and 2), elements that are the result of choices in embodiment: in this case, for example, the teachers' participation in already structured IMEWs (task structures a and b). This aspect of embodiment can be taken up and adapted to other IME contexts, possibly modifying the specific experience, but retaining its fundamental features, such as the use of mathematical artifacts.

10. Conclusion

In summary, this article aimed to shed light on the foundational aspects of the InformalMath project. We started presenting IME (Section 2), as an approach that promotes mathematical engagement in flexible, interdisciplinary environments without traditional grading, empowering learners to freely explore and innovate beyond the confines of structured school curricula. We then presented the InformalMath project, which applies IME principles in non-scientific museums to create dynamic learning experiences that integrate historical artifacts and mathematical inquiry through a laboratory approach (Section 3). In Section 4 we clarified the aim of this work, and we presented in the following sections the InformalMath as it developed as a teacher education programme and as an educational design research (Sections 5 and 6). We chose to refer to the technique of conjecture mapping in section 7 in order to answer, in Sections 8 and 9, to our research questions: using conjecture mapping, we have detailed how the characteristics of the IME teacher education programme influence the expected processes and, consequently, the outcomes of InformalMath. Moreover, we identified foundational elements for developing IME teacher education methodologies within the context of non-scientific museums, thereby laying the groundwork for future experimentation.

Looking forward, the InformalMath project stands as a beacon for future initiatives, signaling the importance of educational frameworks that bridge the gap between formal education and the rich informal teacher education opportunities presented by non-scientific museums. By embracing the complexities and emergent learning opportunities within these environments, we pave a way for an educational paradigm that values creativity, interdisciplinarity, and the active participation of teachers in their own professional development. Overall, the InformalMath Project enhances our comprehension of teacher education within the realm of informal mathematics education working in the direction suggested by Nemirovsky and colleagues (2017). Moreover, it serves as a thoughtful step forward in broadening our perspective on mathematics education. Indeed, the project suggests promising directions for refining teacher education and, consequently, for boosting students' engagement and learning in mathematics via informal and museum-based experiences, though these avenues require further exploration and validation. As we continue to explore these intersections, the insights gained from InformalMath will contribute to the ongoing dialogue on innovation in mathematics education.

References

Anichini, G., Arzarello, F., Ciarrapico, L., & Robutti, O. (2004). *Matematica 2003. Attività didattiche e prove di verifica per un nuovo curricolo di matematica (ciclo* *secondario)* [Mathematics 2003. Teaching activities and testing for a new mathematics curriculum (secondary cycle)]. Matteoni Stampatore.

Arzarello, F., & Robutti, O. (2008). Framing the embodied mind approach within a multimodal paradigm. In English, L. D., & Kirshner, D. (Eds.), *Handbook of International Research in Mathematics Education* (pp.720–749). Routledge https://doi.org/10.4324/9780203930236

Bakker, A. (2018). *Design research in education: A practical guide for early career researchers*. Routledge. https://doi.org/10.4324/9780203701010

Boelens, R., De Wever, B., & McKenney, S. (2020). Conjecture mapping to support vocationally educated adult learners in open-ended tasks. *Journal of the Learning Sciences, 29*(3), 430–470. https://doi.org/10.1080/10508406.2020.1759605

Carotenuto, G., Mellone, M., Sabena, C., & Lattaro, P. (2020). Un progetto di educazione matematica informale per prevenire la dispersione scolastica [An informal mathematics education project to prevent early school leaving]. *Matematica, Cultura e Società – Rivista dell'Unione Matematica Italiana, 5*(2), pp. 157–172.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1982). Na vida dez; na escola zero: os contextos culturais da aprendizagem da matemática [In life ten; in school zero: the cultural contexts of learning mathematics]. *Cadernos de Pesquisa, 42*, 79–86.

Casi, R. (2024). Exploring the design of informal mathematics workshops in art museums from the teachers' perspective. In L. Björklund Boistrup, B. Di Paola (Eds.), *Proceedings of the CIEAEM 74 conference Mathematics and practices: Actions for futures, "Quaderni di Ricerca in Didattica", Numero speciale* (pp. 481–490).

Casi, R., & Sabena, C. (2022). Informal mathematics experiences in museums: What potential for teacher professional development? In Hodgen, J., Geraniou, E., Bolondi, G., & Ferretti, F. (Eds.), *Proceedings of the Twelfth Congress of European Research in Mathematics Education (CERME12)* (pp. 3065–3066). ERME and Free University of Bozen-Bolzano.

Casi, R., Leo, V., Pizzarelli, C., & Sabena, C. (2022). La matematica nei musei con il progetto Next-Land [Mathematics in museums within the Next-Land project]. In E. Luciano, M. Oggero, & C. Sabena (Eds.), *Atti dell'Associazione Subalpina Mathesis 2020-22* (pp. 105–116).

Casi, R., & Sabena, C. (2023). Informal mathematics in teacher's education: The teachers' voice. In Drijvers, P., Csapodi, C., Palmér, H., Gosztonyi, K., & Kónya, E. (Eds.), *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)*. (pp. 3353–3360). Alfréd Rényi Institute of Mathematics and ERME.

Casi, R., & Sabena, C. (2024). Mathematics in art and history museums: An informal mathematics education case for teachers' in-service training. *Education Sciences*, 14(5), Article 489. https://doi.org/10.3390/educsci14050489

Childs, C. P., & Greenfield, P. M. (1980). Informal modes of learning and teaching: The case of Zinacanteco weaving. In N. Warren (Ed.), *Studies in cross-cultural psychology* (Vol. 2, pp. 269–316). Academic Press.

Choppin, J., Amador, J., Carson, C., & Callard, C. (2018). Development and use of a conjecture map for online professional development model. In Hodges, T.E., Roy,

G. J., & Tyminski, A. M. (Eds.), Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 1291-1298). University of South Carolina & Clemson University.

Collins, A., Brown, J. S., & Newman, S. E. (1989). Cognitive apprenticeship: Teaching the crafts of reading, writing, and mathematics. In L.B. Resnick (Ed.), *Knowing, learning, and instruction: Essays in honor of Robert Glaser* (pp. 453–494). Lawrence Erlbaum Associates, Inc.

Deister, F. L., Garzetti, M., & Schlauch, M. (2022). Conjecture maps in practice: Planning, conducting and assessing educational design research. *Bildungsforschung, 2*, 1–16.

Kelton, M. L., & Nemirovsky, R. (2023). Politics and aesthetics of museum mathematics: the dissensual curriculum of early 21st century mathematics exhibitions. *Journal of Curriculum Studies*, *55*(1), 82–104. https://doi.org/10.1080/00220272.2022.2061301

Lave, J. (1977). Cognitive consequences of traditional apprenticeship training in West Africa. *Anthropology & Education Quarterly*, *8*(3), 177–180.

McKenney, S., & Reeves, T. (2019). *Conducting educational design research.* Routledge.

Nemirovsky, R. (2018). Pedagogies of emergent learning. In G. Kaiser, H. Forgasz, M. Graven, A. Kuzniak, E. Simmt, & B. Xu (Eds.), *Invited Lectures from the 13th International Congress on Mathematical Education*. ICME-13 Monographs. Springer. https://doi.org/10.1007/978-3-319-72170-5_23

Nemirovsky, R., Kelton, M. L., & Civil, M. (2017). Toward a vibrant and socially significant informal mathematics education. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 90–101). National Council of Teachers of Mathematics.

Nunes, T., Schliemann, A., & Carraher, D. (1993). *Street mathematics and school mathematics*. Cambridge University Press.

Salomon, G. (1996). Studying novel learning environments as patterns of change. In S., Vosniadou, E., De Corte, R., Glaser, & H., Mandl (Eds.), *International perspectives on the design of technology-supported learning environments* (pp. 363–377). Erlbaum.

Sandoval, W. (2014). Conjecture mapping: An approach to systematic educational design research. *Journal of the learning sciences*, *23*(1), 18–36.

Van den Akker, J. (1999). Principles and methods of development research. In J. van den Akker, R. M. Branch, K. Gustafson, N. Nieveen, & T. Plomp (Eds.), *Design Approaches and Tools in Education and Training* (pp.1–14). Springer.